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# New types of fuzzy ideals in BCK/BCI-algebras

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ABSTRACT

A generalization of an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of a BCK/BCI-algebra is discussed. Characterizations of an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal are provided. Conditions for an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal (resp. $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal) to be a fuzzy ideal are provided. Using the notion of a fuzzy ideal with thresholds, characterizations of a fuzzy ideal, an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal are discussed.



## 1. Introduction

The notion of a fuzzy subset was introduced by Zadeh [1] in 1965. In [2], the idea of *fuzzy point* and its *belongingness to* and *quasi-coincidence with* a fuzzy subset were used to define *$(\alpha, \beta)$-fuzzy subgroups*, where $\alpha, \beta \in \{\in, \mathrm{q}, \in \vee \mathrm{q}, \in \wedge \mathrm{q}\}$ and $\alpha \neq \in \wedge \mathrm{q}$. This was further studied in detail by Bhakat [3,4], Bhakat and Das [5,6], and Yuan et al. [7]. The concept of $(\in, \in \vee \mathrm{q})$-fuzzy subgroup is a viable generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar types of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Jun and Song [8] discussed general forms of fuzzy interior ideals in semigroups. Also, Jun [9,10] introduced the concept of $(\alpha, \beta)$-fuzzy subalgebra of a BCK/BCI-algebra and investigated related results. A generalization of a fuzzy ideal in a BCK/BCI-algebra is discussed by Jun [11] and Guangji et al. [12]. Ma et al. [13] discussed $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy ideals of BCI-algebras. Zhan and Jun [14] dealt with $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative (resp., BCI-implicative, BCI-commutative) ideals in BCI-algebras. Zhan et al. [15] considered $(\in, \in \vee \mathrm{q})$-fuzzy $p$-ideals, $(\in, \in \vee \mathrm{q})$-fuzzy $q$-ideals and $(\in, \in \vee \mathrm{q})$-fuzzy $a$-ideals in BCI-algebras.

In this paper, we try to have more general form of an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of a BCK/BCI-algebra. We introduce the notion of an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal in a BCK/BCI-algebra, and give examples which are $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal but not $(\in, \in \vee \mathrm{q})$-fuzzy ideal. We characterize an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal. We provide conditions for an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal (resp. $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal) to be a fuzzy ideal. Using the notion of a fuzzy ideal with thresholds, we discuss characterizations of a fuzzy ideal, an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal. We finally consider characterizations of a fuzzy ideal, an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal by using implication operators and the notion of implication-based fuzzy ideals. The important achievement of the study with an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal is that the notion of an $(\in, \in \vee \mathrm{q})$-fuzzy ideal is a special case of an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal, and thus so many results in the papers [12,11] are corollaries of our results obtained in this paper.

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## 2. Preliminaries

A BCK-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, \theta)$ of type $(2, 0)$ is called a BCI-*algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = \theta)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = \theta)$,
(III) $(\forall x \in X)\ (x * x = \theta)$,
(IV) $(\forall x, y \in X)\ (x * y = \theta, y * x = \theta \Rightarrow x = y)$.

If a BCI-algebra $X$ satisfies the following identity:

(V) $(\forall x \in X)\ (\theta * x = \theta)$,

then $X$ is called a BCK-*algebra*. Any BCK/BCI-algebra $X$ satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * \theta = x)$,
(a2) $(\forall x, y, z \in X)\ (x * y = \theta \Rightarrow (x * z) * (y * z) = \theta, (z * y) * (z * x) = \theta)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ (((x * z) * (y * z)) * (x * y) = \theta)$.

A non-empty subset $A$ of a BCK/BCI-algebra $X$ is called an *ideal* of $X$, denoted by $A \lhd X$, if it satisfies:

(d1) $\theta \in A$,
(d2) $(\forall x \in X)\ (\forall y \in A)\ (x * y \in A \Rightarrow x \in A)$.

We refer the reader to the books [16,17] for further information regarding BCK/BCI-algebras.

For any fuzzy subset $\mathcal{A}$ of a set $X$ and any $t \in [0, 1]$ the set

$$U(\mathcal{A}; t) = \{x \in X \mid \mathcal{A}(x) \geq t\}$$

is called a *level subset* of $\mathcal{A}$.

A fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$ is called a *fuzzy ideal* of $X$ if it satisfies:

(d3) $(\forall x \in X)\ (\mathcal{A}(\theta) \geq \mathcal{A}(x))$,
(d4) $(\forall x, y \in X)\ (\mathcal{A}(x) \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\})$.

A fuzzy subset $\mathcal{A}$ of a set $X$ of the form

$$\mathcal{A}(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support $x$ and value $t$ and is denoted by $[x; t]$.

For a fuzzy subset $\mathcal{A}$ of a set $X$, we say that a fuzzy point $[x; t]$ is

(d5) *contained* in $\mathcal{A}$, denoted by $[x; t] \in \mathcal{A}$, [2] if $\mathcal{A}(x) \geq t$.
(d6) *quasi-coincident* with $\mathcal{A}$, denoted by $[x; t]\mathrm{q}\mathcal{A}$, [2] if $\mathcal{A}(x) + t > 1$.

For a fuzzy point $[x; t]$ and a fuzzy subset $\mathcal{A}$ of a set $X$, we say that

(d7) $[x; t] \in \vee\mathrm{q}\mathcal{A}$ if $[x; t] \in \mathcal{A}$ or $[x; t]\mathrm{q}\mathcal{A}$.
(d8) $[x; t]\overline{\alpha}\mathcal{A}$ if $[x; t]\alpha\mathcal{A}$ does not hold for $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}\}$.

## 3. New types of fuzzy ideals

Let $k$ denote an arbitrary element of $[0, 1)$ unless otherwise specified. For a fuzzy point $[x; t]$ and a fuzzy subset $\mathcal{A}$ of $X$, we say that

(d9) $[x; t]\mathrm{q}_k\mathcal{A}$ if $\mathcal{A}(x) + t + k > 1$.
(d10) $[x; t] \in \vee\mathrm{q}_k\mathcal{A}$ if $[x; t] \in \mathcal{A}$ or $[x; t]\mathrm{q}_k\mathcal{A}$.
(d11) $[x; t]\overline{\alpha}\mathcal{A}$ if $[x; t]\alpha\mathcal{A}$ does not hold for $\alpha \in \{\mathrm{q}_k, \in\vee\mathrm{q}_k\}$.

The following theorem is a generalization of [11, Theorem 3.2].

**Theorem 3.1.** *Let $\mathcal{A}$ be a fuzzy subset of a BCK/BCI-algebra $X$. Then the following are equivalent:*

(1) $\left(\forall t \in \left(\frac{1-k}{2}, 1\right]\right)\ (U(\mathcal{A}; t) \neq \emptyset \Rightarrow U(\mathcal{A}; t) \lhd X)$.
(2) *$\mathcal{A}$ satisfies the following assertions:*
  (2.1) $(\forall x \in X)\ \left(\mathcal{A}(x) \leq \max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\}\right)$.
  (2.2) $(\forall x, y \in X)\ \left(\min\{\mathcal{A}(x * y), \mathcal{A}(y)\} \leq \max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\}\right)$.

**Table 1**
$*$-multiplication table for $X$.

| $*$ | $\theta$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $\theta$ | $\theta$ | $\theta$ | $\theta$ | $\theta$ | $\theta$ |
| $a$ | $a$ | $\theta$ | $a$ | $\theta$ | $a$ |
| $b$ | $b$ | $b$ | $\theta$ | $b$ | $\theta$ |
| $c$ | $c$ | $a$ | $c$ | $\theta$ | $c$ |
| $d$ | $d$ | $d$ | $b$ | $d$ | $\theta$ |

**Proof.** Assume that (1) is valid. If there is $a \in X$ such that the condition (2.1) is not valid, that is,

$$(\exists a \in X) \quad \left(\mathcal{A}(a) > \max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\}\right),$$

then $\mathcal{A}(a) \in \left(\frac{1-k}{2}, 1\right]$ and $a \in U(\mathcal{A}; \mathcal{A}(a))$. But $\mathcal{A}(\theta) < \mathcal{A}(a)$ implies that $\theta \notin U(\mathcal{A}; \mathcal{A}(a))$, a contradiction. Hence (2.1) is valid. Suppose that (2.2) is false, i.e.,

$$s := \min\{\mathcal{A}(a * b), \mathcal{A}(b)\} > \max\left\{\mathcal{A}(a), \frac{1-k}{2}\right\}$$

for some $a, b \in X$. Then $s \in \left(\frac{1-k}{2}, 1\right]$ and $a * b, b \in U(\mathcal{A}; s)$. But $a \notin U(\mathcal{A}; s)$ since $\mathcal{A}(a) < s$. This is a contradiction, and so (2.2) holds.

Conversely, assume that $\mathcal{A}$ satisfies two conditions (2.1) and (2.2). Let $t \in (\frac{1-k}{2}, 1]$ be such that $U(\mathcal{A}; t) \neq \emptyset$. For any $x \in U(\mathcal{A}; t)$, we have

$$\max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\} \geq \mathcal{A}(x) \geq t > \frac{1-k}{2}$$

and so $\mathcal{A}(\theta) = \max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\} \geq t$. Hence $\theta \in U(\mathcal{A}; t)$. Let $x, y \in X$ be such that $x * y \in U(\mathcal{A}; t)$ and $y \in U(\mathcal{A}; t)$. Then

$$\max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\} \geq t > \frac{1-k}{2}$$

and thus $\mathcal{A}(x) = \max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} \geq t$, i.e., $x \in U(\mathcal{A}; t)$. Therefore $U(\mathcal{A}; t)$ is an ideal of $X$ for all $t \in \left(\frac{1-k}{2}, 1\right]$. $\square$

If we take $k = 0$ in Theorem 3.1, then we have the following corollary.

**Corollary 3.2** ([11, Theorem 3.2]). *Let $\mathcal{A}$ be a fuzzy subset of a BCK/BCI-algebra $X$. Then the following are equivalent:*

(1) $(\forall t \in (0.5, 1])\ (U(\mathcal{A}; t) \neq \emptyset \Rightarrow U(\mathcal{A}; t) \lhd X)$.
(2) *$\mathcal{A}$ satisfies the following assertions:*
  (2.1) $(\forall x \in X)\ (\mathcal{A}(x) \leq \max\{\mathcal{A}(\theta), 0.5\})$.
  (2.2) $(\forall x, y \in X)\ (\min\{\mathcal{A}(x * y), \mathcal{A}(y)\}) \leq \max\{\mathcal{A}(x), 0.5\}$.

**Definition 3.3.** A fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$ is called an $(\in, \in \vee q_k)$-*fuzzy ideal* of $X$ if it satisfies:

(d12) $[x; t] \in \mathcal{A} \Rightarrow [\theta; t] \in \vee q_k \mathcal{A}$.
(d13) $[x * y; t_1] \in \mathcal{A}, [y; t_2] \in \mathcal{A} \Rightarrow [x; \min\{t_1, t_2\}] \in \vee q_k \mathcal{A}$

for all $x, y \in X$ and $t, t_1, t_2 \in (0, 1]$.

An $(\in, \in \vee q_k)$-fuzzy ideal of a BCK/BCI-algebra $X$ with $k = 0$ is called an $(\in, \in \vee q)$-*fuzzy ideal* of $X$.

**Example 3.4.** Let $X = \{\theta, a, b, c, d\}$ be a BCK-algebra with the $*$-multiplication given in Table 1. Let $\mathcal{A}$ be a fuzzy subset of $X$ defined by $\mathcal{A}(\theta) = 0.4$, $\mathcal{A}(a) = \mathcal{A}(c) = 0.7$ and $\mathcal{A}(b) = \mathcal{A}(d) = 0.2$. It is routine to verify that

(1) $\mathcal{A}$ is an $(\in, \in \vee q_{0.4})$-fuzzy ideal of $X$.
(2) $\mathcal{A}$ is not an $(\in, \in \vee q_{0.12})$-fuzzy ideal of $X$.
(3) $\mathcal{A}$ is not a fuzzy ideal of $X$.
(4) $\mathcal{A}$ is not an $(\in, \in \vee q)$-fuzzy ideal of $X$ since $[c; 0.5] \in A$ but $[\theta; 0.5] \overline{\in \vee q} \mathcal{A}$.

**Example 3.5.** Let $X = \{\theta, 1, 2, a, b\}$ be a BCI-algebra where the $*$-multiplication is defined by Table 2. A fuzzy subset $\mathcal{A}$ of $X$ defined by $\mathcal{A}(\theta) = 0.4$, $\mathcal{A}(1) = \mathcal{A}(b) = 0.1$, $\mathcal{A}(2) = 0.5$ and $\mathcal{A}(a) = 0.3$ is an $(\in, \in \vee q_{0.2})$-fuzzy ideal of $X$, but it is not an $(\in, \in \vee q_{0.08})$-fuzzy ideal of $X$ since $\mathcal{A}(\theta) = 0.4 < 0.46 = \min\{\mathcal{A}(2), 0.46\}$.

**Theorem 3.6.** *Let $X$ be a BCK/BCI-algebra. A fuzzy subset $\mathcal{A}$ of $X$ is an $(\in, \in \vee q_k)$-fuzzy ideal of $X$ if and only if it satisfies:*

(1) $(\forall x \in X)\ \left(\mathcal{A}(\theta) \geq \min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\}\right)$,
(2) $(\forall x, y \in X)\ \left(\mathcal{A}(x) \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\}\right)$.

**Table 2**
$*$-multiplication table for $X$.

| $*$ | $\theta$ | 1 | 2 | $a$ | $b$ |
|---|---|---|---|---|---|
| $\theta$ | $\theta$ | $\theta$ | $\theta$ | $a$ | $a$ |
| 1 | 1 | $\theta$ | 1 | $b$ | $a$ |
| 2 | 2 | 2 | $\theta$ | $a$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | $\theta$ | $\theta$ |
| $b$ | $b$ | $a$ | $b$ | 1 | $\theta$ |

**Proof.** Suppose that $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$. Let $x \in X$ and assume that $\mathcal{A}(x) < \frac{1-k}{2}$. If $\mathcal{A}(\theta) < \mathcal{A}(x)$, then $\mathcal{A}(\theta) < t \leq \mathcal{A}(x)$ for some $t \in \left(0, \frac{1-k}{2}\right)$. It follows that $[x; t] \in \mathcal{A}$ but $[\theta; t]\overline{\in}\mathcal{A}$. Since $\mathcal{A}(\theta) + t < 2t < 1 - k$, we get $[\theta; t]\overline{\mathrm{q}_k}\mathcal{A}$. Therefore $[\theta; t]\overline{\in \vee \mathrm{q}_k}\mathcal{A}$, which is a contradiction. Hence $\mathcal{A}(\theta) \geq \mathcal{A}(x)$. Now if $\mathcal{A}(x) \geq \frac{1-k}{2}$ then $\left[x; \frac{1-k}{2}\right] \in \mathcal{A}$ and so $\left[\theta; \frac{1-k}{2}\right] \in \vee \mathrm{q}_k\mathcal{A}$, which implies that $\mathcal{A}(\theta) \geq \frac{1-k}{2}$ or $\mathcal{A}(\theta) + \frac{1-k}{2} > 1 - k$. Hence $\mathcal{A}(\theta) \geq \frac{1-k}{2}$. Otherwise, $\mathcal{A}(\theta) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, a contradiction. Consequently, $\mathcal{A}(\theta) \geq \min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\}$ for all $x \in X$. Let $x, y \in X$ and suppose that

$$\min\{\mathcal{A}(x * y), \mathcal{A}(y)\} < \frac{1-k}{2}.$$

We claim that $\mathcal{A}(x) \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\}$. If not, then

$$\mathcal{A}(x) < t \leq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\}$$

for some $t \in \left(0, \frac{1-k}{2}\right)$. It follows that $[x*y; t] \in \mathcal{A}$ and $[y; t] \in \mathcal{A}$, but $[x; t]\overline{\in}\mathcal{A}$ and $\mathcal{A}(x) + t < 2t < 1-k$, i.e., $[x; t]\overline{\mathrm{q}_k}\mathcal{A}$. This is a contradiction. Thus $\mathcal{A}(x) \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\}$ whenever $\min\{\mathcal{A}(x * y), \mathcal{A}(y)\} < \frac{1-k}{2}$. If $\min\{\mathcal{A}(x * y), \mathcal{A}(y)\} \geq \frac{1-k}{2}$ then $\left[x * y; \frac{1-k}{2}\right] \in \mathcal{A}$ and $\left[y; \frac{1-k}{2}\right] \in \mathcal{A}$. Since $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal, it follows that

$$\left[x; \frac{1-k}{2}\right] = \left[x; \min\left\{\frac{1-k}{2}, \frac{1-k}{2}\right\}\right] \in \vee \mathrm{q}_k\mathcal{A}$$

so that $\mathcal{A}(x) \geq \frac{1-k}{2}$ or $\mathcal{A}(x) + \frac{1-k}{2} > 1 - k$. If $\mathcal{A}(x) < \frac{1-k}{2}$ then

$$\mathcal{A}(x) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$$

which is a contradiction. Therefore $\mathcal{A}(x) \geq \frac{1-k}{2}$. Consequently,

$$\mathcal{A}(x) \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\}$$

for all $x, y \in X$.

Conversely, assume that (1) and (2) are valid. Let $x \in X$ and $t \in (0, 1]$ be such that $[x; t] \in \mathcal{A}$. Then $\mathcal{A}(x) \geq t$. Suppose that $\mathcal{A}(\theta) < t$. If $\mathcal{A}(x) < \frac{1-k}{2}$ then

$$\mathcal{A}(\theta) \geq \min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} = \mathcal{A}(x) \geq t,$$

a contradiction. Hence $\mathcal{A}(x) \geq \frac{1-k}{2}$, which implies that

$$\mathcal{A}(\theta) + t > 2\mathcal{A}(\theta) \geq 2\min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} = 1 - k.$$

Thus $[\theta; t] \in \vee \mathrm{q}_k\mathcal{A}$. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $[x * y; t_1] \in \mathcal{A}$ and $[y; t_2] \in \mathcal{A}$. Then $\mathcal{A}(x * y) \geq t_1$ and $\mathcal{A}(y) \geq t_2$. Suppose that $\mathcal{A}(x) < \min\{t_1, t_2\}$. If $\min\{\mathcal{A}(x * y), \mathcal{A}(y)\} < \frac{1-k}{2}$, then

$$\mathcal{A}(x) \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} = \min\{\mathcal{A}(x * y), \mathcal{A}(y)\} \geq \min\{t_1, t_2\}.$$

This is impossible, and so $\min\{\mathcal{A}(x * y), \mathcal{A}(y)\} \geq \frac{1-k}{2}$. It follows that

$$\mathcal{A}(x) + \min\{t_1, t_2\} > 2\mathcal{A}(x) \geq 2\min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} = 1 - k$$

so that $[x; \min\{t_1, t_2\}] \in \vee \mathrm{q}_k\mathcal{A}$. Therefore $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$. $\square$

If we take $k = 0$ in Theorem 3.6, then we have the following corollary.

**Corollary 3.7** *([12, Theorem 2.3], [11, Theorem 3.7]). Let $X$ be a BCK/BCI-algebra. A fuzzy subset $\mathcal{A}$ of $X$ is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of $X$ if and only if it satisfies:*

(1) $(\forall x \in X)\ (\mathcal{A}(\theta) \geq \min\{\mathcal{A}(x), 0.5\})$,
(2) $(\forall x, y \in X)\ (\mathcal{A}(x) \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y), 0.5\})$.

Obviously, every fuzzy ideal is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal, but the converse may not be true as seen in Examples 3.4 and 3.5.

We give a condition for an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal to be a fuzzy ideal.

**Theorem 3.8.** *Let $\mathcal{A}$ be an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of a BCK/BCI-algebra $X$. If $\mathcal{A}(\theta) < \frac{1-k}{2}$, then $\mathcal{A}$ is a fuzzy ideal of $X$.*

**Proof.** Assume that $\mathcal{A}(\theta) < \frac{1-k}{2}$. Then $\mathcal{A}(x) < \frac{1-k}{2}$ and so $\mathcal{A}(x) \leq \mathcal{A}(\theta) < \frac{1-k}{2}$ for all $x \in X$ by Theorem 3.6(1). It follows from Theorem 3.6(2) that

$$\mathcal{A}(x) \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} = \min\{\mathcal{A}(x * y), \mathcal{A}(y)\}.$$

Hence $\mathcal{A}$ is a fuzzy ideal of $X$. $\square$

**Corollary 3.9** *([12, Theorem 2.5]). Let $\mathcal{A}$ be an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of a BCK/BCI-algebra $X$. If $\mathcal{A}(\theta) < 0.5$, then $\mathcal{A}$ is a fuzzy ideal of $X$.*

**Proof.** It follows from Theorem 3.8 by taking $k = 0$. $\square$

**Theorem 3.10.** *Let $X$ be a BCK/BCI-algebra. If $0 \leq k < r < 1$, then every $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal is an $(\in, \in \vee \mathrm{q}_r)$-fuzzy ideal.*

**Proof.** Straightforward. $\square$

Examples 3.4 and 3.5 show that the converse of Theorem 3.10 is not true.

**Proposition 3.11.** *Every $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal $\mathcal{A}$ of a BCK/BCI-algebra $X$ satisfies the following assertions:*

(1) $(\forall x, y \in X)\ \left(x \leq y \Rightarrow \mathcal{A}(x) \geq \min\left\{\mathcal{A}(y), \frac{1-k}{2}\right\}\right)$.
(2) $(\forall x, y, z \in X)\ \left(x * y \leq z \Rightarrow \mathcal{A}(x) \geq \min\left\{\mathcal{A}(y), \mathcal{A}(z), \frac{1-k}{2}\right\}\right)$.

**Proof.** (1) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = \theta$, and so

$$\begin{aligned}\mathcal{A}(x) &\geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} \\ &= \min\left\{\mathcal{A}(\theta), \mathcal{A}(y), \frac{1-k}{2}\right\} \\ &= \min\left\{\mathcal{A}(y), \frac{1-k}{2}\right\}.\end{aligned}$$

(2) Let $x, y, z \in X$ be such that $x * y \leq z$. Then

$$\begin{aligned}\mathcal{A}(x) &\geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} \\ &\geq \min\left\{\min\left\{\mathcal{A}(z), \frac{1-k}{2}\right\}, \mathcal{A}(y), \frac{1-k}{2}\right\} \\ &= \min\left\{\mathcal{A}(y), \mathcal{A}(z), \frac{1-k}{2}\right\}.\end{aligned}$$

This completes the proof. $\square$

**Corollary 3.12.** *Every $(\in, \in \vee \mathrm{q})$-fuzzy ideal $\mathcal{A}$ of a BCK/BCI-algebra $X$ satisfies the following assertions:*

(1) $(\forall x, y \in X)\ (x \leq y \Rightarrow \mathcal{A}(x) \geq \min\{\mathcal{A}(y), 0.5\})$.
(2) $(\forall x, y, z \in X)\ (x * y \leq z \Rightarrow \mathcal{A}(x) \geq \min\{\mathcal{A}(y), \mathcal{A}(z), 0.5\})$.

**Proof.** It is straightforward by taking $k = 0$ in Proposition 3.11. $\square$

**Theorem 3.13.** *For a fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$, the following are equivalent:*

(1) *$\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$.*
(2) $\left(\forall t \in \left(0, \frac{1-k}{2}\right]\right)\ (U(\mathcal{A}; t) \neq \emptyset \Rightarrow U(\mathcal{A}; t) \lhd X)$.

We say that $U(\mathcal{A}; t)$ is an $(\in \vee \mathrm{q}_k)$-*level ideal* of $\mathcal{A}$ in $X$.

**Proof.** Assume that $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$ and let $t \in \left(0, \frac{1-k}{2}\right]$ be such that $U(\mathcal{A}; t) \neq \emptyset$. Using Theorem 3.6(1), we have

$$\mathcal{A}(\theta) \geq \min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\}$$

for any $x \in U(\mathcal{A}; t)$. It follows that

$$\mathcal{A}(\theta) \geq \min\left\{t, \frac{1-k}{2}\right\} = t$$

so that $\theta \in U(\mathcal{A}; t)$. Let $x, y \in X$ be such that $x * y \in U(\mathcal{A}; t)$ and $y \in U(\mathcal{A}; t)$ for $t \in \left(0, \frac{1-k}{2}\right]$. Then $\mathcal{A}(x * y) \geq t$ and $\mathcal{A}(y) \geq t$. Using Theorem 3.6(2) implies that

$$\mathcal{A}(x) \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t.$$

Thus $x \in U(\mathcal{A}; t)$, and so $U(\mathcal{A}; t)$, $t \in \left(0, \frac{1-k}{2}\right]$, is an ideal of $X$.

Conversely, let $\mathcal{A}$ be a fuzzy subset of $X$ such that $U(\mathcal{A}; t)$ is non-empty and is an ideal of $X$ for all $t \in \left(0, \frac{1-k}{2}\right]$. If there exists $a \in X$ such that $\mathcal{A}(\theta) < \min\left\{\mathcal{A}(a), \frac{1-k}{2}\right\}$, then $\mathcal{A}(\theta) < t_\theta \leq \min\left\{\mathcal{A}(a), \frac{1-k}{2}\right\}$ for some $t_\theta \in \left(0, \frac{1-k}{2}\right]$, and so $\theta \notin U(\mathcal{A}; t_\theta)$. This is a contradiction. Therefore $\mathcal{A}(\theta) \geq \min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\}$ for all $x \in X$. Assume that there exist $a, b \in X$ such that

$$\mathcal{A}(a) < \min\left\{\mathcal{A}(a * b), \mathcal{A}(b), \frac{1-k}{2}\right\}.$$

Then $\mathcal{A}(a) < t_a \leq \min\left\{\mathcal{A}(a * b), \mathcal{A}(b), \frac{1-k}{2}\right\}$ for some $t_a \in \left(0, \frac{1-k}{2}\right]$. It follows that $a * b \in U(\mathcal{A}; t_a)$ and $b \in U(\mathcal{A}; t_a)$, but $a \notin U(\mathcal{A}; t_a)$. This is impossible, and thus

$$\mathcal{A}(x) \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\}$$

for all $x, y \in X$. Using Theorem 3.6, we conclude that $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$. $\square$

Taking $k = 0$ in Theorem 3.13 induces the following corollary.

**Corollary 3.14** (*[12, Theorem 2.6]*). *For a fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$, the following are equivalent:*

(1) *$\mathcal{A}$ is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of $X$.*
(2) $(\forall t \in (0, 0.5])\ (U(\mathcal{A}; t) \neq \emptyset \Rightarrow U(\mathcal{A}; t) \lhd X)$.

For a fuzzy point $[x; t]$ and a fuzzy subset $\mathcal{A}$ of $X$, we say that

(d14) $[x; t]\underline{\mathrm{q}}\mathcal{A}$ if $\mathcal{A}(x) + t \geq 1$,
(d15) $[x; t]\underline{\mathrm{q}}_k\mathcal{A}$ if $\mathcal{A}(x) + t + k \geq 1$.

Denote by $Q(\mathcal{A}; t)$ (resp. $\underline{Q}(\mathcal{A}; t)$) the set $\{x \in X \mid [x; t]\mathrm{q}\mathcal{A}\}$ (resp. $\{x \in X \mid [x; t]\underline{\mathrm{q}}\mathcal{A}\}$), and

$$Q^k(\mathcal{A}; t) := \{x \in X \mid [x; t]\mathrm{q}_k\mathcal{A}\}, \qquad [\mathcal{A}]_t^k := \{x \in X \mid [x; t] \in \vee \mathrm{q}_k\mathcal{A}\}.$$
$$\underline{Q}^k(\mathcal{A}; t) := \{x \in X \mid [x; t]\underline{\mathrm{q}}_k\mathcal{A}\}, \qquad \underline{[\mathcal{A}]}_t^k := \{x \in X \mid [x; t] \in \vee \underline{\mathrm{q}}_k\mathcal{A}\}.$$

Obviously, $[\mathcal{A}]_t^k = U(\mathcal{A}; t) \cup Q^k(\mathcal{A}; t)$ and $\underline{[\mathcal{A}]}_t^k = U(\mathcal{A}; t) \cup \underline{Q}^k(\mathcal{A}; t)$.

**Theorem 3.15.** *If $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$, then*

$$\left(\forall t \in \left(\frac{1-k}{2}, 1\right]\right) \quad (\underline{Q}^k(\mathcal{A}; t) \neq \emptyset \Rightarrow \underline{Q}^k(\mathcal{A}; t) \lhd X).$$

**Proof.** Let $t \in \left(\frac{1-k}{2}, 1\right]$ be such that $\underline{Q}^k(\mathcal{A}; t) \neq \emptyset$. Then there exists $x_0 \in \underline{Q}^k(\mathcal{A}; t)$, and so $\mathcal{A}(x_0) + t \geq 1 - k$. By means of Theorem 3.6(1), we have

$$\mathcal{A}(\theta) \geq \min\left\{\mathcal{A}(x_0), \frac{1-k}{2}\right\} \geq \min\left\{1 - k - t, \frac{1-k}{2}\right\} = 1 - k - t,$$

i.e., $[\theta; t]\underline{\mathrm{q}}_k\mathcal{A}$. Hence $\theta \in \underline{Q}^k(\mathcal{A}; t)$. Let $x, y \in X$ be such that $x * y \in \underline{Q}^k(\mathcal{A}; t)$ and $y \in \underline{Q}^k(\mathcal{A}; t)$. Then $\mathcal{A}(x * y) + t \geq 1 - k$ and $\mathcal{A}(y) + t \geq 1 - k$. It follows from Theorem 3.6(2) that

$$\mathcal{A}(x) \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} \geq \min\left\{1 - k - t, \frac{1-k}{2}\right\} = 1 - k - t$$

so that $[x; t]\underline{\mathrm{q}}_k\mathcal{A}$, i.e., $x \in \underline{Q}^k(\mathcal{A}; t)$. Hence $\underline{Q}^k(\mathcal{A}; t)$ is an ideal of $X$. □

**Corollary 3.16.** *If $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of $X$, then*

$$(\forall t \in (0.5, 1]) \quad (\underline{Q}(\mathcal{A}; t) \neq \emptyset \Rightarrow \underline{Q}(\mathcal{A}; t) \lhd X).$$

**Corollary 3.17.** *Let $\mathcal{A}$ be an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$. If $k < r < 1$, then*

$$\left(\forall t \in \left(\frac{1-r}{2}, 1\right]\right) \quad (\underline{Q}^r(\mathcal{A}; t) \neq \emptyset \Rightarrow \underline{Q}^r(\mathcal{A}; t) \lhd X).$$

**Proof.** It is straightforward by Theorems 3.10 and 3.15. □

**Theorem 3.18.** *For any fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$, the following are equivalent:*

(1) *$\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$.*
(2) $(\forall t \in (0, 1])$ $(\underline{[\mathcal{A}]}_t^k \neq \emptyset \Rightarrow \underline{[\mathcal{A}]}_t^k \lhd X)$.

We call $\underline{[\mathcal{A}]}_t^k$ an $(\in \vee \underline{\mathrm{q}}_k)$-*level ideal* of $\mathcal{A}$.

**Proof.** Assume that $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$ and let $t \in (0, 1]$ such that $\underline{[\mathcal{A}]}_t^k \neq \emptyset$. Then there exists $a \in \underline{[\mathcal{A}]}_t^k$, and so $a \in U(\mathcal{A}; t)$ or $a \in \underline{Q}^k(\mathcal{A}; t)$, i.e., $\mathcal{A}(a) \geq t$ or $\mathcal{A}(a) + t \geq 1 - k$. Using Theorem 3.6(1), we get

$$\mathcal{A}(\theta) \geq \min\left\{\mathcal{A}(a), \frac{1-k}{2}\right\}. \tag{3.1}$$

We consider two cases: $\mathcal{A}(a) \leq \frac{1-k}{2}$ and $\mathcal{A}(a) > \frac{1-k}{2}$. For the first case, we have $\mathcal{A}(\theta) \geq \mathcal{A}(a)$ by (3.1). Thus if $\mathcal{A}(a) \geq t$, then $\mathcal{A}(\theta) \geq t$ and so $\theta \in U(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. If $\mathcal{A}(a) + t \geq 1 - k$, then $\mathcal{A}(\theta) + t \geq \mathcal{A}(a) + t \geq 1 - k$ which implies that $[\theta; t]\underline{\mathrm{q}}_k\mathcal{A}$, i.e., $\theta \in \underline{Q}^k(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. Combining the second case and (3.1) induces $\mathcal{A}(\theta) \geq \frac{1-k}{2}$. If $t \leq \frac{1-k}{2}$, then $\mathcal{A}(\theta) \geq t$ and hence $\theta \in U(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. If $t > \frac{1-k}{2}$, then $\mathcal{A}(\theta) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, which implies that $\theta \in Q^k(\mathcal{A}; t) \subseteq \underline{Q}^k(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. Therefore $\underline{[\mathcal{A}]}_t^k$ satisfies the condition (d1). Let $x, y \in X$ be such that $x * y \in \underline{[\mathcal{A}]}_t^k$ and $y \in \underline{[\mathcal{A}]}_t^k$. Then $x * y \in U(\mathcal{A}; t)$ or $[x * y; t]\underline{\mathrm{q}}_k\mathcal{A}$, and $y \in U(\mathcal{A}; t)$ or $[y; t]\underline{\mathrm{q}}_k\mathcal{A}$, that is, $\mathcal{A}(x * y) \geq t$ or $\mathcal{A}(x * y) + t \geq 1 - k$, and $\mathcal{A}(y) \geq t$ or $\mathcal{A}(y) + t \geq 1 - k$. We consider the following four cases:

(i) $\mathcal{A}(x * y) \geq t$ and $\mathcal{A}(y) \geq t$,
(ii) $\mathcal{A}(x * y) \geq t$ and $\mathcal{A}(y) + t \geq 1 - k$,
(iii) $\mathcal{A}(x * y) + t \geq 1 - k$ and $\mathcal{A}(y) \geq t$,
(iv) $\mathcal{A}(x * y) + t \geq 1 - k$ and $\mathcal{A}(y) + t \geq 1 - k$.

Since $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$, we have

$$\mathcal{A}(x) \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} \tag{3.2}$$

by Theorem 3.6(2). Using (i) and (3.2), we get $\mathcal{A}(x) \geq \min\left\{t, \frac{1-k}{2}\right\}$. If $t \leq \frac{1-k}{2}$, then $\mathcal{A}(x) \geq t$, i.e., $x \in U(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. If $t > \frac{1-k}{2}$, then $\mathcal{A}(x) \geq \frac{1-k}{2}$ and so $\mathcal{A}(x) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$. Hence $x \in Q^k(\mathcal{A}; t) \subseteq \underline{Q}^k(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. Case (ii) and (3.2) imply that $\mathcal{A}(x) \geq \min\left\{t, 1 - k - t, \frac{1-k}{2}\right\}$. If $t \leq \frac{1-k}{2}$, then $\mathcal{A}(x) \geq \min\{t, 1 - k - t\} = t$ and so $x \in U(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. If $t > \frac{1-k}{2}$ then $\mathcal{A}(x) \geq \min\left\{1 - k - t, \frac{1-k}{2}\right\} = 1 - k - t$ and thus $x \in \underline{Q}^k(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. Similarly, we have $x \in \underline{[\mathcal{A}]}_t^k$ from the case (iii) and (3.2). Finally, case (iv) and (3.2) induces $\mathcal{A}(x) \geq \min\left\{1 - k - t, \frac{1-k}{2}\right\}$. If $t \leq \frac{1-k}{2}$, then $\mathcal{A}(x) \geq \frac{1-k}{2} \geq t$. Hence $x \in U(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. If $t > \frac{1-k}{2}$ then $\mathcal{A}(x) \geq 1 - k - t$ which implies that $x \in \underline{Q}^k(\mathcal{A}; t) \subseteq \underline{[\mathcal{A}]}_t^k$. Consequently, $\underline{[\mathcal{A}]}_t^k$ is an ideal of $X$.

Conversely, suppose that (2) is valid. If there exists $a \in X$ such that $\mathcal{A}(\theta) < \min\left\{\mathcal{A}(a), \frac{1-k}{2}\right\}$, then $\mathcal{A}(\theta) < t_\theta \leq \min\left\{\mathcal{A}(a), \frac{1-k}{2}\right\}$ for some $t_\theta \in \left(0, \frac{1-k}{2}\right]$. It follows that $a \in U(\mathcal{A}; t_\theta) \subseteq \underline{[\mathcal{A}]}_{t_\theta}^k$ but $\theta \notin U(\mathcal{A}; t_\theta)$. Also, we have $\mathcal{A}(\theta) + t_\theta <$

$2t_\theta \leq 1-k$, and so $[\theta; t_\theta]\overline{\underline{\mathrm{q}}_k}\mathcal{A}$, i.e., $\theta \notin \underline{Q}^k(\mathcal{A}; t)$. Therefore $\theta \notin \underline{[\mathcal{A}]}^k_{t_\theta}$, a contradiction. Hence $\mathcal{A}(\theta) \geq \min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\}$ for all $x \in X$. Suppose that there exist $a, b \in X$ such that $\mathcal{A}(a) < \min\left\{\mathcal{A}(a*b), \mathcal{A}(b), \frac{1-k}{2}\right\}$. Then

$$\mathcal{A}(a) < t_a \leq \min\left\{\mathcal{A}(a*b), \mathcal{A}(b), \frac{1-k}{2}\right\}$$

for some $t_a \in \left(0, \frac{1-k}{2}\right]$. It follows that $a*b, b \in U(\mathcal{A}; t_a) \subseteq \underline{[\mathcal{A}]}^k_{t_a}$ so from (d2) that $a \in \underline{[\mathcal{A}]}^k_{t_a}$. Thus $\mathcal{A}(a) \geq t_a$ or $\mathcal{A}(a) + t_a \geq 1-k$, a contradiction. Therefore $\mathcal{A}(x) \geq \min\left\{\mathcal{A}(x*y), \mathcal{A}(y), \frac{1-k}{2}\right\}$ for all $x, y \in X$. Using Theorem 3.6, we conclude that $\mathcal{A}$ is an $(\in, \in\vee \mathrm{q}_k)$-fuzzy ideal of $X$. $\square$

**Corollary 3.19.** *For any fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$, the following are equivalent:*
(1) *$\mathcal{A}$ is an $(\in, \in\vee \mathrm{q})$-fuzzy ideal of $X$,*
(2) $(\forall t \in (0,1])\ (\underline{[\mathcal{A}]}_t \neq \emptyset \Rightarrow \underline{[\mathcal{A}]}_t \lhd X)$,
*where $\underline{[\mathcal{A}]}_t := \{x \in X \mid [x; t] \in \vee \mathrm{q}\mathcal{A}\} = U(\mathcal{A}; t) \cup \underline{Q}(\mathcal{A}; t)$.*

A fuzzy subset $\mathcal{A}$ of $X$ is said to be *proper* if $\mathrm{Im}(\mathcal{A})$ has at least two elements. Two fuzzy subsets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

**Theorem 3.20.** *Let $\mathcal{A}$ be an $(\in, \in\vee \mathrm{q}_k)$-fuzzy ideal of a BCK/BCI-algebra $X$ such that $\#\left\{\mathcal{A}(x) \mid \mathcal{A}(x) < \frac{1-k}{2}\right\} \geq 2$. Then there exist two proper non-equivalent $(\in, \in\vee \mathrm{q}_k)$-fuzzy ideals of $X$ such that $\mathcal{A}$ can be expressed as the union of them.*

**Proof.** Let $\left\{\mathcal{A}(x) \mid \mathcal{A}(x) < \frac{1-k}{2}\right\} = \{t_1, t_2, \ldots, t_n\}$, where $t_1 > t_2 > \cdots > t_n$ and $n \geq 2$. Then the chain of $\in\vee \underline{\mathrm{q}}_k$-level ideals of $\mathcal{A}$ is

$$\underline{[\mathcal{A}]}^k_{\frac{1-k}{2}} \subseteq \underline{[\mathcal{A}]}^k_{t_1} \subseteq \underline{[\mathcal{A}]}^k_{t_2} \subseteq \cdots \subseteq \underline{[\mathcal{A}]}^k_{t_n} = X.$$

Let $\mathcal{B}$ and $\mathcal{C}$ be fuzzy subsets of $X$ defined by

$$\mathcal{B}(x) = \begin{cases} t_1 & \text{if } x \in \underline{[\mathcal{A}]}^k_{t_1}, \\ t_2 & \text{if } x \in \underline{[\mathcal{A}]}^k_{t_2} \setminus \underline{[\mathcal{A}]}^k_{t_1}, \\ \cdots & \\ t_n & \text{if } x \in \underline{[\mathcal{A}]}^k_{t_n} \setminus \underline{[\mathcal{A}]}^k_{t_{n-1}}, \end{cases}$$

and

$$\mathcal{C}(x) = \begin{cases} \mathcal{A}(x) & \text{if } x \in \underline{[\mathcal{A}]}^k_{\frac{1-k}{2}}, \\ k & \text{if } x \in \underline{[\mathcal{A}]}^k_{t_2} \setminus \underline{[\mathcal{A}]}^k_{\frac{1-k}{2}}, \\ t_3 & \text{if } x \in \underline{[\mathcal{A}]}^k_{t_3} \setminus \underline{[\mathcal{A}]}^k_{t_2}, \\ \cdots & \\ t_n & \text{if } x \in \underline{[\mathcal{A}]}^k_{t_n} \setminus \underline{[\mathcal{A}]}^k_{t_{n-1}}, \end{cases}$$

respectively, where $t_3 < k < t_2$. Then $\mathcal{B}$ and $\mathcal{C}$ are $(\in, \in\vee \mathrm{q}_k)$-fuzzy ideals of $X$, and $\mathcal{B}, \mathcal{C} \leq \mathcal{A}$. The chains of $\in\vee \underline{\mathrm{q}}_k$-level ideals of $\mathcal{B}$ and $\mathcal{C}$ are, respectively, given by

$$\underline{[\mathcal{A}]}^k_{t_1} \subseteq \underline{[\mathcal{A}]}^k_{t_2} \subseteq \cdots \subseteq \underline{[\mathcal{A}]}^k_{t_n}$$

and

$$\underline{[\mathcal{A}]}^k_{\frac{1-k}{2}} \subseteq \underline{[\mathcal{A}]}^k_{t_2} \subseteq \cdots \subseteq \underline{[\mathcal{A}]}^k_{t_n}.$$

Therefore $\mathcal{B}$ and $\mathcal{C}$ are non-equivalent and clearly $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$. This completes the proof. $\square$

**Theorem 3.21.** *Let $\{\mathcal{A}_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee \mathrm{q}_k)$-fuzzy ideals of a BCK/BCI-algebra $X$. Then $\mathcal{A} := \bigcap_{i\in\Lambda} \mathcal{A}_i$ is an $(\in, \in\vee \mathrm{q}_k)$-fuzzy ideal of $X$.*

**Proof.** Let $x \in X$ and $t \in (0,1]$ be such that $[x; t] \in \mathcal{A}$. Assume that $[\theta; t]\overline{\in\vee \mathrm{q}_k}\mathcal{A}$. Then $\mathcal{A}(\theta) < t$ and $\mathcal{A}(\theta) + t \leq 1-k$, which imply that

$$\mathcal{A}(\theta) < \frac{1-k}{2}. \tag{3.3}$$

Let $\Omega_1 := \{i \in \Lambda \mid \mathcal{A}_i(\theta) \geq t\}$ and

$$\Omega_2 := \{i \in \Lambda \mid [\theta; t]\mathrm{q}_k\mathcal{A}_i \text{ and } \mathcal{A}_i(\theta) < t\}.$$

Then $\Lambda = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. If $\Omega_2 = \emptyset$, then $\mathcal{A}_i(\theta) \geq t$ for all $i \in \Lambda$, and so $\mathcal{A}(\theta) \geq t$ which is a contradiction. Hence $\Omega_2 \neq \emptyset$, and so $\mathcal{A}_i(\theta) + t > 1-k$ and $\mathcal{A}_i(\theta) < t$ for every $i \in \Omega_2$. It follows that $t > \frac{1-k}{2}$ so that $\mathcal{A}_i(x) \geq \mathcal{A}(x) \geq t > \frac{1-k}{2}$

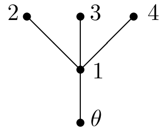

**Fig. 1.**

for all $i \in \Lambda$. Now, suppose that $t_\theta := \mathcal{A}_i(\theta) < \frac{1-k}{2}$ for some $i \in \Lambda$. Let $t'_\theta \in \left(0, \frac{1-k}{2}\right)$ be such that $t_\theta < t'_\theta$. Then $\mathcal{A}_i(x) > \frac{1-k}{2} > t'_\theta$, i.e., $[x; t'_\theta] \in \mathcal{A}_i$. But $\mathcal{A}_i(\theta) = t_\theta < t'_\theta$ and $\mathcal{A}_i(\theta) + t'_\theta < 1 - k$, that is, $[\theta; t'_\theta]\overline{\in \vee \mathrm{q}_k}\mathcal{A}_i$. This is a contradiction, and so $\mathcal{A}_i(\theta) \geq \frac{1-k}{2}$ for all $i \in \Lambda$. Thus $\mathcal{A}(\theta) \geq \frac{1-k}{2}$, which is impossible. Therefore $[\theta; t] \in \vee \mathrm{q}_k\mathcal{A}$. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $[x * y; t_1] \in \mathcal{A}$ and $[y; t_2] \in \mathcal{A}$. Assume that $[x; \min\{t_1, t_2\}]\overline{\in \vee \mathrm{q}_k}\mathcal{A}$. Then $\mathcal{A}(x) < \min\{t_1, t_2\}$ and $\mathcal{A}(x) + \min\{t_1, t_2\} \leq 1 - k$. It follows that $\mathcal{A}(x) < \frac{1-k}{2}$. Let $\Omega_3 := \{i \in \Lambda \mid \mathcal{A}_i(x) \geq \min\{t_1, t_2\}\}$ and

$$\Omega_4 := \{i \in \Lambda \mid [x; \min\{t_1, t_2\}]\mathrm{q}_k\mathcal{A}_i \text{ and } \mathcal{A}_i(x) < \min\{t_1, t_2\}\}.$$

Then $\Lambda = \Omega_3 \cup \Omega_4$ and $\Omega_3 \cap \Omega_4 = \emptyset$. If $\Omega_4 = \emptyset$, then $\mathcal{A}_i(x) \geq \min\{t_1, t_2\}$ for all $i \in \Lambda$, and so $\mathcal{A}(x) \geq \min\{t_1, t_2\}$ which is a contradiction. Hence $\Omega_4 \neq \emptyset$, and thus $[x; \min\{t_1, t_2\}]\mathrm{q}_k\mathcal{A}_i$, i.e., $\mathcal{A}_i(x) + \min\{t_1, t_2\} > 1 - k$, and $\mathcal{A}_i(x) < \min\{t_1, t_2\}$. It follows that $\min\{t_1, t_2\} > \frac{1-k}{2}$ so that

$$\mathcal{A}_i(x * y) \geq \mathcal{A}(x * y) \geq t_1 \geq \min\{t_1, t_2\} > \frac{1-k}{2}$$

for all $i \in \Lambda$. Similarly, we have $\mathcal{A}_i(y) > \frac{1-k}{2}$ for all $i \in \Lambda$. Now, suppose that $t := \mathcal{A}_i(x) < \frac{1-k}{2}$ for some $i \in \Lambda$. Let $t' \in \left(0, \frac{1-k}{2}\right)$ be such that $t < t'$. Then $\mathcal{A}_i(x * y) > \frac{1-k}{2} > t'$ and $\mathcal{A}_i(y) > \frac{1-k}{2} > t'$, that is, $[x * y; t'] \in \mathcal{A}_i$ and $[y; t'] \in \mathcal{A}_i$. But $\mathcal{A}_i(x) = t < t'$ and $\mathcal{A}_i(x) + t' < 1 - k$, that is, $[x; t']\overline{\in \vee \mathrm{q}_k}\mathcal{A}_i$. This is a contradiction, and hence $\mathcal{A}_i(x) \geq \frac{1-k}{2}$ for all $i \in \Lambda$. Therefore $\mathcal{A}(x) \geq \frac{1-k}{2}$, which is invalid. Consequently, $[x; \min\{t_1, t_2\}] \in \vee \mathrm{q}_k\mathcal{A}$. Accordingly, $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$. □

Taking $k = 0$ in Theorem 3.21, we have the following corollary.

**Corollary 3.22.** *Let $\{\mathcal{A}_i \mid i \in \Lambda\}$ be a family of $(\in, \in \vee \mathrm{q})$-fuzzy ideals of a BCK/BCI-algebra $X$. Then $\mathcal{A} := \bigcap_{i\in\Lambda} \mathcal{A}_i$ is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of $X$.*

The following example shows that there exists $k \in [0, 1)$ such that the union of two $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideals of a BCK/BCI-algebra $X$ may not be an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$.

**Example 3.23.** Consider the BCI-algebra $X = \{\theta, 1, 2, a, b\}$ and the $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal $\mathcal{A}$ of $X$ which are established in Example 3.5. Let $\mathcal{B}$ be a fuzzy subset of $X$ defined by

$$\mathcal{B} = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0.7 & 0.6 & 0.5 & 0.2 & 0.2 \end{pmatrix}.$$

It can be easily verified that $\mathcal{B}$ is an $(\in, \in \vee \mathrm{q}_{0.2})$-fuzzy ideal of $X$. But, $\mathcal{A} \cup \mathcal{B}$ is not an $(\in, \in \vee \mathrm{q}_{0.2})$-fuzzy ideal of $X$ since $[b * 1; 0.28] \in \mathcal{A} \cup \mathcal{B}$ and $[1; 0.56] \in \mathcal{A} \cup \mathcal{B}$, however $[b; \min\{0.28, 0.56\}] = [b; 0.28]\overline{\in \vee \mathrm{q}_{0.2}}\mathcal{A} \cup \mathcal{B}$.

It is well known that a fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$ is a fuzzy ideal of $X$ if and only if the non-empty level subset $U(\mathcal{A}; t)$, $t \in (0, 1]$, of $\mathcal{A}$ is an ideal of $X$. Note that for a fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$, the non-empty level subset $U(\mathcal{A}; t)$, $t \in (0, \frac{1-k}{2}]$, of $\mathcal{A}$ is an ideal of $X$ if and only if $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$ (see Theorem 3.13).

Since it is natural to consider the number $t \in \left(\frac{1-k}{2}, 1\right]$ for which $U(\mathcal{A}; t)$ is an ideal of $X$, we consider a new kind of a fuzzy ideal as follows.

**Definition 3.24.** A fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$ is called an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-*fuzzy ideal* of $X$ if it satisfies:

(d16) $[\theta; t]\overline{\in}\mathcal{A} \Rightarrow [x; t]\overline{\in} \vee \overline{\mathrm{q}_k}\mathcal{A}$,
(d17) $[x; \min\{t_1, t_2\}]\overline{\in}\, \mathcal{A} \Rightarrow [x * y; t_1]\overline{\in} \vee \overline{\mathrm{q}_k}\mathcal{A}$or $[y; t_2]\overline{\in} \vee \overline{\mathrm{q}_k}\mathcal{A}$

for all $x, y \in X$ and $t, t_1, t_2 \in (0, 1]$.

An $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal of a BCK/BCI-algebra $X$ with $k = 0$ is called an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}})$-*fuzzy ideal* of $X$.

**Example 3.25.** Consider a set $X = \{\theta, 1, 2, 3, 4\}$. The following Hasse diagram (Fig. 1) makes $X$ into a BCK-algebra where the BCK-operation $*$ is given by

$$x * y := \begin{cases} \theta & \text{if } x \leq y, \\ x & \text{if } y = \theta, \\ 1 & \text{if } x = 2 \text{ and } y \notin \{\theta, 2\}, \\ 3 & \text{if } x = 3 \text{ and } y \neq 3, \\ 4 & \text{if } x = 4 \text{ and } y \neq 4 \end{cases}$$

for every $x, y \in X$. Let $\mathcal{A}$ be a fuzzy subset of $X$ defined by

$$\mathcal{A} = \begin{pmatrix} \theta & 1 & 2 & 3 & 4 \\ 0.8 & 0.7 & 0.7 & 0.2 & 0.5 \end{pmatrix}.$$

It is routine to verify that $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.7}})$-fuzzy ideal of $X$.

Obviously, every fuzzy ideal is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal, but the converse may not be true as seen in the following example.

**Example 3.26.** Consider the BCI-algebra which is given in Example 3.5. Let $\mathcal{A}$ be a fuzzy subset of $X$ defined by

$$\mathcal{A} = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0.8 & 0.3 & 0.6 & 0.5 & 0.2 \end{pmatrix}.$$

It is routine to check that $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.08}})$-fuzzy ideal of $X$, but it is not a fuzzy ideal of $X$ since

$$\mathcal{A}(b) = 0.2 < 0.3 = \min\{\mathcal{A}(b * a), \mathcal{A}(a)\}.$$

Let $\mathcal{A}$ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of a BCK/BCI-algebra $X$. Suppose that there exists $a \in X$ such that $\mathcal{A}(a) > \max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\}$. Then $\mathcal{A}(a) \geq t > \max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\}$ for some $t \in \left(\frac{1-k}{2}, 1\right]$. It follows that $[\theta; t]\overline{\in}\mathcal{A}$, $[a; t] \in \mathcal{A}$ and $\mathcal{A}(a) + t \geq 2t > 1 - k$, i.e., $[a; t]q_k\mathcal{A}$. This is a contradiction, and so the following inequality is valid.

(e1) $(\forall x \in X)\ \left(\mathcal{A}(x) \leq \max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\}\right)$.

Now, suppose that $\max\left\{\mathcal{A}(a), \frac{1-k}{2}\right\} < \min\{\mathcal{A}(a * b), \mathcal{A}(b)\}$ for some $a, b \in X$. Then there exists $t \in \left(\frac{1-k}{2}, 1\right]$ such that

$$\max\left\{\mathcal{A}(a), \frac{1-k}{2}\right\} < t \leq \min\{\mathcal{A}(a * b), \mathcal{A}(b)\}.$$

Thus $[a; t]\overline{\in}\mathcal{A}$. From $t \leq \min\{\mathcal{A}(a*b), \mathcal{A}(b)\}$, we have $[a*b; t] \in \mathcal{A}$, $[b; t] \in \mathcal{A}$, $\mathcal{A}(a*b)+t \geq 2t > 1-k$, i.e., $[a*b; t]q_k\mathcal{A}$, and $\mathcal{A}(b) + t \geq 2t > 1 - k$, i.e., $[b; t]q_k\mathcal{A}$. This is impossible, and hence we know that $\mathcal{A}$ satisfies the following assertion:

(e2) $(\forall x, y \in X)\ \left(\max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\}\right)$.

Let $\mathcal{A}$ be a fuzzy subset of a BCK/BCI-algebra $X$ satisfying (e1) and (e2). Let $t \in \left(\frac{1-k}{2}, 1\right]$ be such that $U(\mathcal{A}; t) \neq \emptyset$. Then there exists $a \in U(\mathcal{A}; t)$, and so

$$\frac{1-k}{2} < t \leq \mathcal{A}(a) \leq \max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\} = \mathcal{A}(\theta)$$

by (e1). Hence $\theta \in U(\mathcal{A}; t)$. Let $x, y \in X$ be such that $x * y \in U(\mathcal{A}; t)$ and $y \in U(\mathcal{A}; t)$. Then $\mathcal{A}(x * y) \geq t > \frac{1-k}{2}$ and $\mathcal{A}(y) \geq t > \frac{1-k}{2}$. Using (e2), we get

$$\max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\} \geq t > \frac{1-k}{2}$$

which implies that $\mathcal{A}(x) = \max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} \geq t$. Thus $x \in U(\mathcal{A}; t)$. Consequently, $U(\mathcal{A}; t) \lhd X$. Therefore we conclude that if a fuzzy subset $\mathcal{A}$ of $X$ satisfies two conditions (e1) and (e2), then the following assertion is valid.

(e3) $\left(\forall t \in \left(\frac{1-k}{2}, 1\right]\right)\ (U(\mathcal{A}; t) \neq \emptyset \Rightarrow U(\mathcal{A}; t) \lhd X)$.

Now, let $\mathcal{A}$ be a fuzzy subset of a BCK/BCI-algebra $X$ satisfying (e3). Let $x \in X$ and $t \in (0, 1]$ be such that $[x; t]\overline{\in \vee q_k}\mathcal{A}$. Then $[x; t] \in \mathcal{A}$ and $[x; t]q_k\mathcal{A}$. Hence $x \in U(\mathcal{A}; t)$, i.e., $U(\mathcal{A}; t) \neq \emptyset$, and so $U(\mathcal{A}; t) \lhd X$ by (e3). Thus $\theta \in U(\mathcal{A}; t)$, and thus $\mathcal{A}(\theta) \geq t$, i.e., $[\theta; t] \in \mathcal{A}$. This shows that (d16) is valid. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $[x * y; t_1]\overline{\in \vee q_k}\mathcal{A}$ and $[y; t_2]\overline{\in \vee q_k}\mathcal{A}$. Then $[x * y; t_1] \in A$, $[y; t_2] \in A$, $[x * y; t_1]q_k\mathcal{A}$ and $[y; t_2]q_k\mathcal{A}$, which imply that $x * y \in U(\mathcal{A}; t_1) \subseteq U(\mathcal{A}; \min\{t_1, t_2\})$ and $y \in U(\mathcal{A}; t_2) \subseteq U(\mathcal{A}; \min\{t_1, t_2\})$. Since $U(\mathcal{A}; \min\{t_1, t_2\}) \lhd X$ by (e3), it follows from (d2) that $x \in U(\mathcal{A}; \min\{t_1, t_2\})$, i.e., $\mathcal{A}(x) \geq \min\{t_1, t_2\}$ so that $[x; \min\{t_1, t_2\}] \in \mathcal{A}$. Hence (d17) is valid, and so $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of $X$. Therefore we have the following theorem.

**Theorem 3.27.** *For a fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$, the following are equivalent:*

(1) *$\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of $X$.*
(2) *$\mathcal{A}$ satisfies the condition* (e3).
(3) *$\mathcal{A}$ satisfies two conditions* (e1) *and* (e2).

**Corollary 3.28** ([12, Theorems 2.8 and 2.9]). *For a fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra $X$, the following are equivalent:*

(1) *$\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}})$-fuzzy ideal of $X$.*
(2) $(\forall t \in (0.5, 1])\ (U(\mathcal{A}; t) \neq \emptyset \Rightarrow U(\mathcal{A}; t) \lhd X)$.
(3) *$\mathcal{A}$ satisfies the following conditions:*
  (i) $(\forall x \in X)\ (\mathcal{A}(x) \leq \max\{\mathcal{A}(\theta), 0.5\})$.
  (ii) $(\forall x, y \in X)\ (\max\{\mathcal{A}(x), 0.5\} \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\})$.

For a fuzzy subset $\mathcal{A}$ of $X$, we consider the following set:

$$\Gamma := \{t \in (0, 1] \mid U(\mathcal{A}; t) \neq \emptyset \Rightarrow U(\mathcal{A}; t) \lhd X\}.$$

Then

(1) If $\Gamma = [0, 1]$, then $\mathcal{A}$ is a fuzzy ideal of $X$.
(2) If $\Gamma = \left(0, \frac{1-k}{2}\right]$, then $\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$.
(3) If $\Gamma = \left(\frac{1-k}{2}, 1\right]$, then $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal of $X$.

Now we have the following question:

**Question.** If $\Gamma = (\varepsilon, \delta]$ where $\varepsilon < \delta$ in $(0, 1]$, then what kind of a fuzzy ideal is $\mathcal{A}$? and what is the relation between them?

To discuss this question, we introduce the following definition.

**Definition 3.29.** A fuzzy subset $\mathcal{A}$ of $X$ is called a *fuzzy ideal with thresholds $\varepsilon$ and $\delta$* of $X$, where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$, if it satisfies the following conditions:

(d18) $(\forall x \in X)\ (\max\{\mathcal{A}(\theta), \varepsilon\} \geq \min\{\mathcal{A}(x), \delta\})$.
(d19) $(\forall x, y \in X)\ (\max\{\mathcal{A}(x), \varepsilon\} \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y), \delta\})$.

**Example 3.30.** Consider a BCI-algebra $X = \{\theta, 1, 2, a, b\}$ with a $*$-multiplication given by Table 2. Let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers and let $Y = X \times \mathbb{Z}$ be the direct product of $X$ and $\mathbb{Z}$. Then $Y$ is a BCI-algebra with the zero element $(\theta, 0)$. For every $n \in \mathbb{Z}$, define a fuzzy subset $\mathcal{A}$ of $Y$ by

$$\mathcal{A} = \begin{pmatrix} (\theta, n) & (1, n) & (2, n) & (a, n) & (b, n) \\ 0.7 & 0.7 & 0.9 & 0.5 & 0.3 \end{pmatrix}.$$

It is easy to check that $\mathcal{A}$ is a fuzzy ideal of $Y$ with thresholds $\varepsilon = 0.6$ and $\delta = 0.7$. However, it is not a fuzzy ideal of $Y$ with thresholds $\varepsilon = 0.6$ and $\delta = 0.8$ since $\max\{\mathcal{A}(\theta, n), 0.6\} = 0.7 \ngeq 0.8 = \min\{\mathcal{A}(2, n), 0.8\}$. If we take $\varepsilon = 0.4$ and $\delta = 0.5$, then $\mathcal{A}$ satisfies the condition (d18). But it does not satisfy the condition (d19) since

$$\max\{\mathcal{A}(b, n), 0.4\} = 0.4 \ngeq 0.5 = \min\{\mathcal{A}(b * a, n), \mathcal{A}(a, n), 0.5\}.$$

Hence it is not a fuzzy ideal of $Y$ with thresholds $\varepsilon = 0.4$ and $\delta = 0.5$.

We provide a characterization of fuzzy ideal with thresholds.

**Theorem 3.31.** *Let $\mathcal{A}$ be a fuzzy subset of $X$ and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then $\mathcal{A}$ is a fuzzy ideal with thresholds $\varepsilon$ and $\delta$ of $X$ if and only if it satisfies:*

$$(\forall t \in (\varepsilon, \delta]) \quad (U(\mathcal{A}; t) \neq \emptyset \Rightarrow U(\mathcal{A}; t) \lhd X). \tag{3.4}$$

**Proof.** The proof is similar to the proof of Theorems 3.13 and 3.27. □

Note that every fuzzy ideal is a fuzzy ideal with some thresholds, but the converse may not be true. In fact, the fuzzy subset $\mathcal{A}$ in Example 3.30 is a fuzzy ideal of $X$ with thresholds $\varepsilon = 0.6$ and $\delta = 0.7$, but it is not a fuzzy ideal of $X$ since $\mathcal{A}(\theta, n) < \mathcal{A}(2, n)$.

The following example shows that there exist $\varepsilon, \delta \in (0, 1]$ with $\varepsilon < \delta$ such that $\mathcal{A}$ is a fuzzy ideal with thresholds $\varepsilon$ and $\delta$ which is not an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal.

**Example 3.32.** Consider a BCI-algebra $X = \{\theta, 1, 2, a, b\}$ in Example 3.5. The fuzzy subset $\mathcal{A}$ of $X$ which is given in Example 3.5 is not an $(\in, \in \vee \mathrm{q}_{0.08})$-fuzzy ideal of $X$. We can easily check that $\mathcal{A}$ is a fuzzy ideal of $X$ with thresholds $\varepsilon = 0.3$ and $\delta = 0.4$.

The following example shows that there exist $\varepsilon, \delta \in (0, 1]$ with $\varepsilon < \delta$ such that $\mathcal{A}$ is a fuzzy ideal with thresholds $\varepsilon$ and $\delta$ which is not an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal.

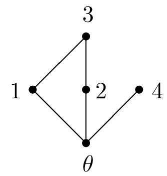


**Fig. 2.**

**Example 3.33.** Consider a set $X = \{\theta, 1, 2, 3, 4\}$. The following Hasse diagram (Fig. 2) makes $X$ into a BCK-algebra where the BCK-operation $*$ is given as follows: $3 * 1 = 2$, $3 * 2 = 1$ and

$$x * y := \begin{cases} \theta & \text{if } x \leq y, \\ x & \text{if } y = \theta, \text{ or } x \text{ and } y \text{ are incomparable.} \end{cases}$$

Let $\mathcal{A}$ be a fuzzy subset of $X$ defined by

$$\mathcal{A} = \begin{pmatrix} \theta & 1 & 2 & 3 & 4 \\ 0.8 & 0.5 & 0.7 & 0.4 & 0.2 \end{pmatrix}.$$

Then $\mathcal{A}$ is a fuzzy ideal of $X$ with thresholds $\varepsilon = 0.2$ and $\delta = 0.4$. If we take $k = 0.4$, then $\max\{\mathcal{A}(3), 0.3\} < \min\{\mathcal{A}(3 * 2), \mathcal{A}(2)\}$. Hence $\mathcal{A}$ is not an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_{0.4}})$-fuzzy ideal of $X$.

**Theorem 3.34.** *Let $\mathcal{A}$ be a fuzzy subset of $X$ and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then*

(1) *$\mathcal{A}$ is a fuzzy ideal of $X$ if and only if $\mathcal{A}$ is a fuzzy ideal of $X$ with thresholds $\varepsilon = 0$ and $\delta = 1$.*
(2) *$\mathcal{A}$ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy ideal of $X$ if and only if $\mathcal{A}$ is a fuzzy ideal of $X$ with thresholds $\varepsilon = 0$ and $\delta = \frac{1-k}{2}$.*
(3) *$\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal of $X$ if and only if $\mathcal{A}$ is a fuzzy ideal of $X$ with thresholds $\varepsilon = \frac{1-k}{2}$ and $\delta = 1$.*

**Proof.** Straightforward. □

## 4. Implication-based fuzzy ideals

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition $\Phi$ is denoted by $[\Phi]$. For a universe $U$ of discourse, we display the fuzzy logical and corresponding set theoretical notations used in this paper

$$[x \in \mathcal{A}] = \mathcal{A}(x), \tag{4.1}$$
$$[\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\}, \tag{4.2}$$
$$[\Phi \rightarrow \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}, \tag{4.3}$$
$$[\forall x \Phi(x)] = \inf_{x \in U} [\Phi(x)], \tag{4.4}$$
$$\models \Phi \quad \text{if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{4.5}$$

The truth valuation rules given in (4.3) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines–Rescher implication operator ($I_{\mathrm{GR}}$):

$$I_{\mathrm{GR}}(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator ($I_{\mathrm{G}}$):

$$I_{\mathrm{G}}(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

(c) The contraposition of Gödel implication operator ($\bar{I}_{\mathrm{G}}$):

$$\bar{I}_{\mathrm{G}}(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ 1 - a & \text{otherwise.} \end{cases}$$

Ying [18] introduced the concept of fuzzifying topology. We can expand his/her idea to BCK/BCI-algebras, and we define a fuzzifying ideal as follows.

**Definition 4.1.** A fuzzy subset $\mathcal{A}$ of $X$ is called a *fuzzifying ideal* of $X$ if it satisfies the following conditions:

(d20) $(\forall x \in X)\ (\models [x \in \mathcal{A}] \to [\theta \in \mathcal{A}])$,
(d21) $(\forall x, y \in X)\ (\models [x * y \in \mathcal{A}] \wedge [y \in \mathcal{A}] \to [x \in \mathcal{A}])$.

Obviously, conditions (d20) and (d21) are equivalent to (d3) and (d4), respectively. Therefore a fuzzifying ideal is an ordinary fuzzy ideal. In [19], the concept of $t$-tautology is introduced, i.e.,

$$\models_t \Phi \quad \text{if and only if } [\Phi] \geq t \text{ for all valuations.} \tag{4.6}$$

**Definition 4.2.** Let $\mathcal{A}$ be a fuzzy subset of $X$ and $t \in (0, 1]$. $\mathcal{A}$ is called a *$t$-implication-based fuzzy ideal* of $X$ if it satisfies:

(d22) $(\forall x \in X)\ (\models_t [x \in \mathcal{A}] \to [\theta \in \mathcal{A}])$,
(d23) $(\forall x, y \in X)\ (\models_t [x * y \in \mathcal{A}] \wedge [y \in \mathcal{A}] \to [x \in \mathcal{A}])$.

Let $I$ be an implication operator. Clearly, $\mathcal{A}$ is a $t$-implication-based fuzzy ideal of $X$ if and only if it satisfies

(d24) $(\forall x \in X)\ (I(\mathcal{A}(x), \mathcal{A}(\theta)) \geq t)$,
(d25) $(\forall x, y \in X)\ (I(\min\{\mathcal{A}(x * y), \mathcal{A}(y)\}, \mathcal{A}(x)) \geq t)$.

**Theorem 4.3.** *For any fuzzy subset $\mathcal{A}$ of $X$, we have*

(1) *if $I = I_{GR}$, then $\mathcal{A}$ is a 0.5-implication-based fuzzy ideal of $X$ if and only if $\mathcal{A}$ is a fuzzy ideal of $X$.*
(2) *If $I = I_G$, then $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy ideal of $X$ if and only if $\mathcal{A}$ is an $(\in, \in \vee q_k)$-fuzzy ideal of $X$.*
(3) *If $I = \bar{I}_G$, then $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy ideal of $X$ if and only if $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal $X$.*

**Proof.** (1) Straightforward.
(2) Assume that $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy ideal of $X$. Then $I_G(\mathcal{A}(x), \mathcal{A}(\theta)) \geq \frac{1-k}{2}$ and $I_G(\min\{\mathcal{A}(x * y), \mathcal{A}(y)\}, \mathcal{A}(x)) \geq \frac{1-k}{2}$. It follows that $\mathcal{A}(\theta) \geq \mathcal{A}(x)$ or $\mathcal{A}(x) \geq \mathcal{A}(\theta) \geq \frac{1-k}{2}$, and $\mathcal{A}(x) \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\}$ or $\min\{\mathcal{A}(x * y), \mathcal{A}(y)\} \geq \mathcal{A}(x) \geq \frac{1-k}{2}$. Hence

$$\max\{\mathcal{A}(\theta), 0\} = \mathcal{A}(\theta) \geq \min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\}$$

and

$$\max\{\mathcal{A}(x), 0\} = \mathcal{A}(x) \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\}.$$

Therefore $\mathcal{A}$ is a fuzzy ideal of $X$ with thresholds $\varepsilon = 0$ and $\delta = \frac{1-k}{2}$, and hence $\mathcal{A}$ is an $(\in, \in \vee q_k)$-fuzzy ideal of $X$ by Theorem 3.34.

Conversely, suppose that $\mathcal{A}$ is an $(\in, \in \vee q_k)$-fuzzy ideal of $X$. Then

$$\mathcal{A}(\theta) = \max\{\mathcal{A}(\theta), 0\} \geq \min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\}$$

and

$$\mathcal{A}(x) = \max\{\mathcal{A}(x), 0\} \geq \min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\}.$$

For the first case, if $\min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} = \mathcal{A}(x)$ then

$$I_G(\mathcal{A}(x), \mathcal{A}(\theta)) = 1 \geq \frac{1-k}{2}.$$

If $\min\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} = \frac{1-k}{2}$ then $\mathcal{A}(\theta) \geq \frac{1-k}{2}$ and so $I_G(\mathcal{A}(x), \mathcal{A}(\theta)) \geq \frac{1-k}{2}$. For the second case, if $\min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} = \min\{\mathcal{A}(x * y), \mathcal{A}(y)\}$ then $\mathcal{A}(x) \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\}$ and thus

$$I_G(\min\{\mathcal{A}(x * y), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 \geq \frac{1-k}{2}.$$

Suppose that $\min\left\{\mathcal{A}(x * y), \mathcal{A}(y), \frac{1-k}{2}\right\} = \frac{1-k}{2}$. Then $\mathcal{A}(x) \geq \frac{1-k}{2}$, and hence

$$I_G(\min\{\mathcal{A}(x * y), \mathcal{A}(y)\}, \mathcal{A}(x)) \geq \frac{1-k}{2}.$$

Therefore $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy ideal of $X$.

(3) Suppose that $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal $X$. Then $\mathcal{A}$ is a fuzzy ideal of $X$ with thresholds $\varepsilon = \frac{1-k}{2}$ and $\delta = 1$ by Theorem 3.34. Thus

$$\max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\} \geq \min\{\mathcal{A}(x), 1\}$$

and

$$\max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} \geq \min\{\mathcal{A}(x*y), \mathcal{A}(y), 1\}.$$

For the first case, if $\mathcal{A}(x) = 1$ then $\max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\} = 1$ and thus

$$\bar{I}_G(\mathcal{A}(x), \mathcal{A}(\theta)) = 1 \geq \frac{1-k}{2}.$$

If $\mathcal{A}(x) < 1$, then $\max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\} \geq \mathcal{A}(x)$. Thus, if $\max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\} = \mathcal{A}(\theta)$ then $\mathcal{A}(\theta) \geq \mathcal{A}(x)$ and so

$$\bar{I}_G(\mathcal{A}(x), \mathcal{A}(\theta)) = 1 \geq \frac{1-k}{2}.$$

If $\max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\} = \frac{1-k}{2}$, then $\mathcal{A}(x) \leq \frac{1-k}{2}$ which implies that

$$\bar{I}_G(\mathcal{A}(x), \mathcal{A}(\theta)) = 1 \geq \frac{1-k}{2}$$

when $\mathcal{A}(\theta) \geq \mathcal{A}(x)$; and

$$\bar{I}_G(\mathcal{A}(x), \mathcal{A}(\theta)) = 1 - \mathcal{A}(x) \geq \frac{1-k}{2}$$

when $\mathcal{A}(\theta) < \mathcal{A}(x)$. For the second case, if $\min\{\mathcal{A}(x*y), \mathcal{A}(y), 1\} = 1$, then

$$\bar{I}_G(\min\{\mathcal{A}(x*y), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 \geq \frac{1-k}{2}.$$

Assume that $\min\{\mathcal{A}(x*y), \mathcal{A}(y), 1\} = \min\{\mathcal{A}(x*y), \mathcal{A}(y)\}$. Then

$$\max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} \geq \min\{\mathcal{A}(x*y), \mathcal{A}(y)\}.$$

If $\max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} = \mathcal{A}(x)$, then $\mathcal{A}(x) \geq \min\{\mathcal{A}(x*y), \mathcal{A}(y)\}$ and so

$$\bar{I}_G(\min\{\mathcal{A}(x*y), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 \geq \frac{1-k}{2}.$$

If $\max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} = \frac{1-k}{2}$ then $\mathcal{A}(x) \leq \frac{1-k}{2}$ and $\min\{\mathcal{A}(x*y), \mathcal{A}(y)\} \leq \frac{1-k}{2}$. Hence

$$\bar{I}_G(\min\{\mathcal{A}(x*y), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 \geq \frac{1-k}{2}$$

when $\mathcal{A}(x) \geq \min\{\mathcal{A}(x*y), \mathcal{A}(y)\}$; and

$$\bar{I}_G(\min\{\mathcal{A}(x*y), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 - \min\{\mathcal{A}(x*y), \mathcal{A}(y)\} \geq \frac{1-k}{2}$$

when $\mathcal{A}(x) < \min\{\mathcal{A}(x*y), \mathcal{A}(y)\}$. Consequently, $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy ideal of $X$.

Conversely, suppose that $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy ideal of $X$. Then $\bar{I}_G(\mathcal{A}(x), \mathcal{A}(\theta)) \geq \frac{1-k}{2}$ and $\bar{I}_G(\min\{\mathcal{A}(x*y), \mathcal{A}(y)\}, \mathcal{A}(x)) \geq \frac{1-k}{2}$ for all $x, y \in X$. It follows that $\mathcal{A}(x) \leq \mathcal{A}(\theta)$ or $1 - \mathcal{A}(x) \geq \frac{1-k}{2}$, i.e., $\mathcal{A}(x) \leq \frac{1-k}{2}$; and $\min\{\mathcal{A}(x*y), \mathcal{A}(y)\} \leq \mathcal{A}(x)$ or $\min\{\mathcal{A}(x*y), \mathcal{A}(y)\} \leq \frac{1-k}{2}$. Thus

$$\max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\} \geq \mathcal{A}(x) = \min\{\mathcal{A}(x), 1\}$$

and

$$\max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\} \geq \min\{\mathcal{A}(x*y), \mathcal{A}(y)\} = \min\{\mathcal{A}(x*y), \mathcal{A}(y), 1\}.$$

Hence $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy ideal of $X$ by Theorem 3.34(3). □

## 5. Conclusion

To obtain a general type of an $(\in, \in \vee q)$-fuzzy ideal of a BCK/BCI-algebra, we have introduced the notion of an $(\in, \in \vee q_k)$-fuzzy ideal. We have provided examples which are $(\in, \in \vee q_k)$-fuzzy ideal but not $(\in, \in \vee q)$-fuzzy ideal. We have dealt with characterizations of an $(\in, \in \vee q_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal. We have investigated conditions for an $(\in, \in \vee q_k)$-fuzzy ideal (resp. $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal) to be a fuzzy ideal. Using the notion of a fuzzy ideal with thresholds, we have discussed characterizations of a fuzzy ideal, an $(\in, \in \vee q_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal. We finally have considered characterizations of a fuzzy ideal, an $(\in, \in \vee q_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal by using implication operators and the notion of implication-based fuzzy ideals.

Work is ongoing. Some important issues for future work are: (1) To develop strategies for obtaining more valuable results, (2) To apply these definitions and results for studying related notions in other fuzzy algebraic structures such as fuzzy rings, fuzzy lattices, fuzzy BCK/BCI-algebras, fuzzy BL-algebras, fuzzy $R_0$-algebras, fuzzy MV-algebras and fuzzy MTL-algebras, etc., (3) To study (fuzzy) soft set theoretical aspects based on these notions herein.


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